CURVES OF MINIMUM MOMENT OF INERTIA WITH RESPECT TO A POINT

By Max Mason

The problem to be considered in the following pages furnishes an example of the calculus of variations in which discontinuous solutions play an important part. The problem may be stated as follows:

Among the curves that join two given points, 1 *and* 2, *that one is required which has the least moment of inertia with respect to a third point* 0.*

It is evident that only those curves which lie in the plane 012 need be considered. In fact, the orthogonal projection, C', on this plane of any curve C not lying in the plane has a smaller moment of inertia than the curve C, since C is longer than C' and each point of C is farther from 0 than is the corresponding point of C'.

Let the polar coordinates of the points 1, 2 be (r_1, θ_1), (r_2, θ_2), the point 0 being chosen as the pole. Assuming the equation of the curve in parameter representation,

$$r = r(t), \theta = \theta(t),$$

it is required to minimize the integral

$$J = \int r^2 ds = \int_{t_1}^{t_2} r^2 \sqrt{r'^2 + r^2\theta'^2}\, dt,$$

under the conditions

$$r(t_1) = r_1, \quad r(t_2) = r_2, \quad \theta(t_1) = \theta_1, \quad \theta(t_2) = \theta_2.$$

The equation of the extremals† may be found most simply by writing the integrand of J in the form

$$r^2 \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr.$$

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* The density is supposed constant and the same for all curves. The problem has not been previously treated beyond the point of determining the equation of the extremals. This was done by Bonnet, *Journal de Mathématiques*, vol. 9 (1844), p. 97.

† For the notation and well known theorems of the calculus of variations the reader is referred to Bolza, *Lectures on the Calculus of Variations*, Chicago, 1904.

Since this does not contain θ we may write down at once a first integral of the differential equation of the extremals, namely,

$$\frac{r^4 \dfrac{d\theta}{dr}}{\sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \text{constant} = \beta,$$

or,

$$\frac{d\theta}{dr} = \frac{\beta}{r\sqrt{r^6 - \beta^2}}.$$

For $\beta = 0$ the extremals are straight lines through the pole, $\theta = \text{const.}$ For general values of β the integration of the above equation gives

$$3\theta + a = \text{arc tan}\, \frac{\sqrt{r^6 - \beta}}{\beta},$$

or,

(1) $$\beta r^3 = \sec\,(3\theta + a),$$

where a is a second constant of integration. An extremal $\theta = \text{const}$ will be denoted by S, the remaining extremals by C.

1. The straight line solutions. In case the three points 0, 1, 2 lie in a straight line the problem may be solved without reference to the general theory. The extremal joining 1 and 2 is the straight line S through 0, 1, 2. Suppose first that the points 1 and 2 are not separated by 0. Let the order be 0, 1, 2. Denote by $\bar{C}$ any curve other than S which joins 1 and 2, and by s the length of arc on $\bar{C}$ or S measured from 2. Now

$$J = \int r^2 ds\,;$$

the length of $\bar{C}$ is greater than that of S, and the value of r on $\bar{C}$ is greater than the value of r on S for the same value of s. Hence $J(\bar{C}) > J(S)$. If the points 1 and 2 are separated by 0 there is a point 3 on S such that the lengths of S_{23} and S_{13} are greater than those of $\bar{C}_{20}$ and $\bar{C}_{10}$ respectively. Then the above inequality holds for the same reason as before.

We may therefore state the following theorem:

THEOREM I. *If the three points* 0, 1, 2 *lie on a straight line, the segment of that line between* 1 *and* 2 *has the least moment of inertia with respect to* 0 *of all curves joining* 1 *and* 2.

2. The Continuous Solutions. It is evident from equation (1) that unless $|\theta_2 - \theta_1| < \pi/3$ the points 1 and 2 cannot be joined by an extremal. We will assume for the present that this inequality holds. Since the extremal of the set (1) corresponding to the values α, β is identical with that for the values $\alpha + \pi$, $-\beta$ we may assume without restriction that

$$0 \leqq \alpha < \pi.$$

To determine an extremal joining 1 and 2, the equations

$$\beta r_1^3 = \sec(3\theta_1 + \alpha)$$

$$\beta r_2^3 = \sec(3\theta_2 + \alpha)$$

must be solved for α and β. The value of β is uniquely determined when α is known. Eliminating β we have

$$r_2^3(\cos 3\theta_2 \cos \alpha - \sin 3\theta_2 \sin \alpha) = r_1^3(\cos 3\theta_1 \cos \alpha - \sin 3\theta_1 \sin \alpha),$$

or,

$$\tan \alpha = \frac{r_1^3 \cos 3\theta_1 - r_2^3 \cos 3\theta_2}{r_1^3 \sin 3\theta_1 - r_2^3 \sin 3\theta_2}.$$

Since $0 \leqq \alpha < \pi$ this equation admits a unique solution. Hence:

If $|\theta_2 - \theta_1| < \pi/3$ *the points* $1(r_1, \theta_1)$, $2(r_2, \theta_2)$ *may be connected by one and only one extremal* C.

Let α_0, β_0 be the values of α, β which determine the extremal C. If we allow β to vary we obtain a one-parameter family of extremals, all having the same pair of radii vectores as asymptotes. These extremals form a field. In fact, the set being given by the equation

$$r^3 = \frac{\sec(3\theta + \alpha_0)}{\beta},$$

we have

$$\frac{\partial r}{\partial \beta} = -\frac{\sec(3\theta + \alpha_0)}{3r^2\beta^2} = -\frac{r}{3\beta},$$

which is different from zero within the angle formed by the asymptotes.

The extremal C *may therefore be imbedded in a field which covers the entire interior of the area bounded by the asymptotes to* C.

Since the function

$$F_1 = \frac{F_{\theta'\theta'}}{r'^2} = \frac{r^4}{(r'^2 + r^2\theta'^2)^{3/2}}$$

is positive for all directions (r', θ') it follows the Weierstrass E-function, $E(r, \theta, r', \theta', \bar{r}', \bar{\theta}')$, is never negative in the field, and is only then zero, when the direction $(\bar{r}', \bar{\theta}')$ coincides with the direction (r', θ'). *Weierstrass's condition is therefore satisfied.*

The sufficient conditions being thus satisfied the extremal C_{12} minimizes the integral with respect to all curves $\bar{C}_{12}$ which lie within the field.

If the comparison curve $\bar{C}_{12}$ does not lie entirely within the field two cases may arise; the closed curve $C_{12} + \bar{C}_{21}$ formed by $\bar{C}_{21}$ and C_{12} may or may not include the point 0. If it does not include 0, the curve $\bar{C}_{12}$ may be replaced by a curve C^*_{12} formed by the curves S_{13}, $\bar{C}_{34}$, S_{42} (figure 1) which by theorem

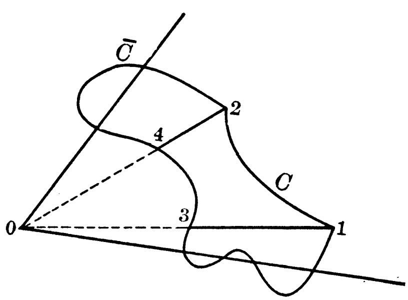

FIG. 1.

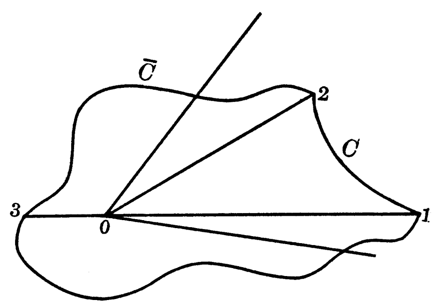

FIG. 2.

I gives a smaller value to J than the value $J(\bar{C}_{12})$. But the curve C^*_{12} lies entirely within the field; hence

$$J(C_{12}) < J(C^*_{12}) < J(\bar{C}_{12}).$$

If $\bar{C}_{21} + C_{12}$ does include the point 0 then the line S_{10} if extended will meet $\bar{C}_{12}$ at a point 3, lying at or beyond 0 on S_{10} (figure 2). Then by theorem I:

$$J(S_{10}) + J(S_{03}) < J(\bar{C}_{13}),$$

$$J(S_{02}) < J(S_{03}) + J(\bar{C}_{32}),$$

Hence, adding these inequalities:

$$J(S_{102}) < J(\bar{C}_{12}).$$

It remains to be proved that

$$J(C_{12}) < J(S_{102}).$$

Let 3, 4 be two points on S_{10} and S_{02} respectively at a distance ϵ from 0. Denote by $C^*_{1\epsilon 2}$ the curve formed by S_{13}, the straight line joining 3 to 4, and S_{42}. Since this curve lies entirely in the field we have

$$J(C^*_{1\epsilon 2}) - J(C_{12}) = \int_{C^*_{1\epsilon 2}} E\,dt,$$

where E is the Weierstrass E function. Hence

$$J(S_{102}) - J(C_{12}) = \operatorname*{limit}_{\epsilon=0} \int_{C^*_{1\epsilon 2}} E\,dt > 0.$$

The results may be stated in the following theorem:

THEOREM II. *The extremal*

$$\beta r^3 = \sec(3\theta + a)$$

joining the points $1(r_1, \theta_1)$, $2(r_2, \theta_2)$, *where* $|\theta_2 - \theta_1| < \pi/3$, *has the least moment of inertia of all curves joining* 1 *and* 2.

3. The Discontinuous Solutions. If $|\theta_2 - \theta_1| \geqq \pi/3$ the points 1 and 2 can not be joined by an extremal having a continuously turning tangent. Any two points may however be joined in an infinite number of ways by a curve composed of portions of different extremals whose intersections are corners of the curve. A necessary condition that such a curve minimize the integral is that

$$F_1(r, \theta, r', \theta') = \frac{r^4}{(r'^2 + r^2\theta'^2)}$$

shall vanish for some direction (r', θ') when the coordinates of a corner are substituted for r and θ. For the problem under consideration this can only happen at the point 0. The only curve which could form a discontinuous solution is therefore the curve S_{102} composed of the lines $\theta = \theta_1$ and $\theta = \theta_2$ which connect 1 and 2 and meet at 0.

It has been seen above that this curve cannot yield an absolute minimum in case $|\theta_2 - \theta_1| < \pi/3$. Moreover it does not give even a relative minimum to J. In fact, any two points $1'$, $2'$ on S_{10} and S_{02} respectively may be joiued

by an extremal C however near $1'$ and $2'$ may be to 0. As has been shown above

$$J(C_{1'2'}) < J(S_{1'02'}),$$

and hence the curve composed of the segments $S_{11'}$, $C_{1'2'}$, $S_{2'2}$, which may lie as near as we please to S_{102}, gives a smaller value to J than S_{102}. *Hence, if* $|\theta_2 - \theta_1| < \pi/3$, *there exists no discontinuous solution.*

It remains to investigate the existence of discontinuous solutions for the case $|\theta_2 - \theta_1| \geqq \pi/3$.

The value of J taken along an extremal C between two points (r_1, θ_1), (r_2, θ_2) for which $|\theta_2 - \theta_1| < \pi/3$ is

$$J(C_{12}) = \int_{r_0}^{r_1} r^2 \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr + \int_{r_0}^{r_2} r^2 \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr,$$

where r_0 is the minimum value of r on the extremal C_{12}. On substituting the value of $\frac{d\theta}{dr}$ from equation (1) this becomes

$$J(C_{12}) = \int_{r_0}^{r_1} \frac{r^5 dr}{\sqrt{r^6 - \beta^{-2}}} + \int_{r_0}^{r_2} \frac{r^5 dr}{\sqrt{r^6 - \beta^{-2}}}.$$

If the points 1 and 2 move so that $|\theta_2 - \theta_1|$ approaches $\pi/3$, then β increases without limit and r_0 approaches 0. Hence

$$\underset{|\theta_2 - \theta_1| = \pi/3}{\text{limit}} J(C_{12}) = \frac{r_1^3 + r_2^3}{3}.$$

But

$$J(S_{102}) = \int_0^{r_1} r^2 dr + \int_0^{r_2} r^2 dr = \frac{r_1^3 + r_2^3}{3},$$

and therefore

$$\underset{|\theta_2 - \theta_1| = \pi/3}{\text{limit}} \{ J(C_{12}) - J(S_{102}) \} = 0. \tag{2}$$

We may now compare the value $J(S_{102})$, for the case $|\theta_2 - \theta_1| = \pi/3$, with the value $J(\bar{C}_{12})$ of the integral taken along any other curve $\bar{C}_{12}$ joining 1 and 2. If $\bar{C}$ does not cross the acute angle formed by the lines $\theta = \theta_1$, $\theta = \theta_2$ then $J(S_{102}) < J(\bar{C}_{12})$. This may be easily shown by the method used in proving

theorem I. Suppose that $\bar{C}$ crosses the angle, intersecting S_{10} at 3 and S_{02} at 4 (see figure 3). By theorem I,

$$J(\bar{C}_{13}) + J(\bar{C}_{42}) \geqq J(S_{13}) + J(S_{42}), \tag{3}$$

the equality holding only when $\bar{C}$ coincides with S_{10} from 1 to 3 and with S_{02} from 4 to 2. Let $3'$ and $4'$ be two points on $\bar{C}$ near 3 and 4 respectively

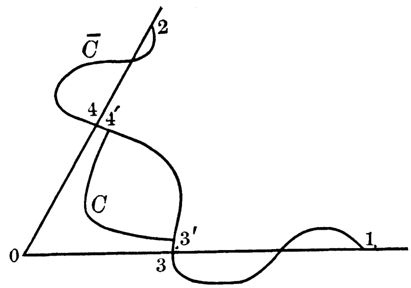

FIG. 3.

and within the angle formed by S_{102}. Then $3'$ and $4'$ may be joined by an extremal C, and

$$J(\bar{C}_{3'4'}) - J(C_{3'4'}) = \int_{\bar{C}_{3'4'}} E\,dt.$$

Hence, by equation (2):

$$J(\bar{C}_{34}) - J(S_{304}) = \underset{3'=3,\ 4'=4}{\text{limit}} \int_{\bar{C}_{3'4'}} E\,dt > 0.$$

Adding this inequality to (3) we have

$$J(\bar{C}_{12}) > J(S_{102}).$$

The curve S_{102} therefore minimizes the integral with respect to all other curves joining 1 and 2, if $|\theta_2 - \theta_1| = \pi/3$.

Suppose now that the end points 1, 2 are so situated that $|\theta_2 - \theta_1| > \pi/3$. It will be shown that the curve S_{102} is the solution in this case.

Let $\bar{C}_{12}$ be any other curve connecting 1 and 2, and $3(r_3, \theta_3)$ a point on $\bar{C}_{12}$ such that $|\theta_3 - \theta_1| = \pi/3$. From the preceding case we have

$$J(\bar{C}_{12}) > J(S_{103}) + J(\bar{C}_{32}),$$

or,

$$J(\bar{C}_{12}) > \frac{r_1^3 + r_3^3}{3} + J(\bar{C}_{32}).$$

Now

$$J(\bar{C}_{32}) = \int_3^2 r^2 \sqrt{r'^2 + r^2 \theta'^2}\, dt \geqq \int_3^2 r^2 |r'|\, dt,$$

$$J(\bar{C}_{32}) \geqq \left| \int_3^2 r^2 r' dt \right| = \frac{|r_2^3 - r_3^3|}{3} \geqq \frac{r_2^3 - r_3^3}{3}.$$

Hence

$$J(\bar{C}_{12}) > \frac{r_1^3 + r_2^3}{3} = J(S_{102}).$$

We may therefore state the theorem:

THEOREM III. *If* $|\theta_2 - \theta_1| \geqq \pi/3$ *the curve* S_{102}, *formed by the straight lines* $\theta = \theta_1$, $\theta = \theta_2$, *has the least moment of inertia with respect to the point* 0 *of all curves connecting the points* 1 *and* 2.

SHEFFIELD SCIENTIFIC SCHOOL OF YALE UNIVERSITY,
NEW HAVEN, CONN.

TRIANGLES AND QUADRILATERALS INSCRIBED TO A CUBIC AND CIRCUMSCRIBED TO A CONIC

BY H. S. WHITE

PONCELET'S porism of inscribed and circumscribed polygons has a certain extension to the system of a non-singular plane cubic and a curve of the second class, resembling that recently given by Morley to point cubic and line cubic, but differing also in one feature. I shall consider only polygons of three and of four sides, first as to their possible occurrence, then by reviewing a well known construction, and lastly as to the corresponding algebraic conditions.